

A hydrodynamic approach to boost invariant free streaming

E. Calzetta*

*Departamento de Física, Facultad de Ciencias Exactas y Naturales,
Universidad de Buenos Aires and IFIBA, CONICET,
Ciudad Universitaria, Buenos Aires 1428, Argentina*

We consider a family of exact boost invariant solutions of the transport equation for free streaming massless particles, where the one particle distribution function is defined in terms of a function of a single variable. The evolution of second and third moments of the one particle distribution function (the second moment being the energy momentum tensor (EMT) and the third moment the non equilibrium current (NEC)) depends only on two moments of that function. For every choice of those two moments we show how to build a non linear hydrodynamic theory which reproduces the exact evolution of the EMT and the NEC. Crude approximations to this theory describe correctly the early time evolution. The structure of these theories may give insight on nonlinear hydrodynamic phenomena on short time scales.

PACS numbers: 52.27.Ny, 52.35.-g, 47.75.+f, 25.75.-q

I. INTRODUCTION

The mounting evidence that relativistic heavy ion collisions [1–6] provide an experimental realization of relativistic real fluids [7] has spurred a strong interest on the characterization of those systems, and as a result we now have a fairly robust model of their dynamics in the limit of short relaxation times [8–18]. However, this very success highlights our lack of understanding of how such a relativistic fluid comes into being in the first place [19–21]. To tackle that issue we need to consider phenomena on time scales which are no longer very large compared with the relaxation time, of which the non linear evolution of plasma instabilities [22–30], shocks [31–38], Kolmogorov [39, 40] and wave turbulence [41, 42] are paramount. Maybe contrary to expectations, it has been found that the language of hydrodynamics is still useful in this regime [43, 44].

If unrealistic, the assumption of free streaming, namely an infinite relaxation time, makes sense methodologically, because in this limit we have simple exact solutions to the kinetic theory (reduced to just the Liouville equation for the one particle distribution function (1PDF)) which provide a test bench for hydrodynamics. A particular set of solutions which has received much attention are boost invariant, axially symmetric 1PDFs described by a function of a single variable and parametrized by a few (Milne) time dependent parameters [45]. In this case it has been shown that energy momentum conservation and a second conservation law derived from the third moment of the Liouville equation determine the time evolution of those parameters, matching exactly the evolution prescribed by the full Liouville equation. This observation has led to the development of so-called anisotropic hydrodynamics [46–51].

If one now considers (color) perturbations of this boost invariant background, then the usual plasma instabilities are found. This formalism may be used to follow those instabilities and to estimate at what time the linearized theory breaks down [52–55]. However, to the present author’s understanding it cannot be pursued beyond that point because we do not have a full nonlinear theory whereby we could compute the back reaction of the unstable modes on the expanding background. The goal of this paper is to investigate which shape such a nonlinear hydrodynamics may have (see also [56]).

To this end we shall consider theories patterned on Geroch - Lindblom “divergence type” hydrodynamics (DTT)[57–63]. The equations of motion are given by the conservation laws for energy momentum tensor (EMT) $T^{\mu\nu}$ and for a third order tensor, the so-called non equilibrium current (NEC) $A^{\mu\nu\rho}$ [64]. We shall consider a conformal colorless fluid, so there will be no conservation law for charge or particle number. The degrees of freedom of the theory are a inverse temperature four vector β^μ and a second order nonequilibrium tensor $\zeta^{\mu\nu}$. A most important assumption is that there is a Massieu function current Φ^μ such that $T^{\mu\nu}$ and $A^{\mu\nu\rho}$ may be obtained as derivatives of Φ^μ with respect to β^μ and $\zeta^{\mu\nu}$. This assumption is key to give the approach predictive power. The hydrodynamic formalism is linked to kinetic theory by relating $T^{\mu\nu}$ and $A^{\mu\nu\rho}$ to the second and third moments of the 1PDF [65–69].

We shall match those theories against a family of exact boost invariant solutions of the Liouville equations. For these solutions, $T^{\mu\nu}$ and $A^{\mu\nu\rho}$ are determined by just two moments of the 1PDF. For every choice of those two

*Electronic address: calzetta@df.uba.ar

parameters, we show how a fully nonlinear hydrodynamic theory may be build that reproduces the known evolution of $T^{\mu\nu}$ and $A^{\mu\nu\rho}$. The resulting theories are probably too complex to be relevant in practice. However, we also show a simple theory which captures the early time behavior, and may be useful to investigate nonlinear effects in this regime.

The rest of the paper is organized as follows. In next section we present the boost invariant, axisymmetric free streaming solutions and compute the EMT and the NEC. Then in Section III we derive the most general hydrodynamic theory where these currents may be derived as derivatives of a single vector potential. Finally in Section IV we match hydrodynamics to the exact solution. We first show that one can always find a potential such that the EMT and NEC derived from solutions to the hydrodynamic equations match the exact solutions. However, since these theories are generally too complex, we show how a drastic simplification of the hydrodynamics still captures the early time behavior. Most importantly, the pressure anisotropy in the simplified hydrodynamics is non decreasing, and so these theories are a suitable framework for the further analysis is isotropization driven by interactions.

We conclude with some brief remarks.

II. BOOST INVARIANT FREE STREAMING

We consider a situation where the particles live in 3+1 dimensions but the 1PDF is independent of the “transverse” coordinates x and y . For simplicity we shall assume axial symmetry in the transverse plane. We will use both cartesian coordinates with interval

$$ds^2 = -dt^2 + dz^2 + dx^2 + dy^2 \quad (1)$$

and Milne coordinates $t = \tau \cosh \chi$, $z = \tau \sinh \chi$, whereby

$$ds^2 = -d\tau^2 + \tau^2 d\chi^2 + dx^2 + dy^2 \quad (2)$$

In the absence of collisions the transport equation reduces to Liouville’s

$$\left\{ p^0 \frac{\partial}{\partial t} + p^z \frac{\partial}{\partial z} \right\} f = 0 \quad (3)$$

We consider a massless theory, so

$$p^0 = \sqrt{p^{z2} + p_{\perp}^2} \quad (4)$$

The most general solution of the transport equation is

$$f = f_0 [p^z t - p^0 z, \mathbf{p}_{\perp}] \quad (5)$$

where f_0 may be any function. Introducing the rapidity Y through $p^0 = p_{\perp} \cosh Y$, $p^z = p_{\perp} \sinh Y$ we see that

$$f = f_0 [\tau p_{\perp} \sinh (Y - \chi), \mathbf{p}_{\perp}] \quad (6)$$

On the other hand, the relationships

$$\begin{aligned} p^0 &= \cosh \chi p^{\tau} + \tau \sinh \chi p^{\chi} \\ p^z &= \sinh \chi p^{\tau} + \tau \cosh \chi p^{\chi} \end{aligned} \quad (7)$$

may be inverted to yield

$$\begin{aligned} p^{\tau} &= p_{\perp} \cosh (Y - \chi) \\ p^{\chi} &= \frac{p_{\perp}}{\tau} \sinh (Y - \chi) \end{aligned} \quad (8)$$

Therefore we see that the general solution to the Liouville equation is

$$f = f_0 [\tau^2 p^\chi, \mathbf{p}_\perp] \quad (9)$$

and is boost invariant if we adopt \mathbf{p}_\perp and p^χ as independent variables.

A. Moments of the free streaming 1PDF

We consider a solution of the type eq. (9), we only assume f_0 is even in p^χ and axially symmetric in the transverse plane. The covariant moments are

$$A_n^{\mu_1, \dots, \mu_n} = \tau \int \frac{dp^\chi d^2 \mathbf{p}_\perp}{(2\pi)^3 p^0} p^{\mu_1} \dots p^{\mu_n} f_0 \quad (10)$$

They are totally symmetric, traceless on any two indexes, and obey conservation laws

$$\nabla_{\mu_n} A_n^{\mu_1, \dots, \mu_n} = I_{n-1}^{\mu_1, \dots, \mu_{n-1}} \quad (11)$$

where

$$I_n^{\mu_1, \dots, \mu_n} = \frac{\tau}{\tau_R} \int \frac{dp^\chi d^2 \mathbf{p}_\perp}{(2\pi)^3 p^0} p^{\mu_1} \dots p^{\mu_n} I_{col} [f_0] \quad (12)$$

We are interested in $A_2^{\mu\nu} = T^{\mu\nu}$, $A_3^{\mu\nu\rho} = A^{\mu\nu\rho}$ and their sources $I_1^\mu = 0$ and $I_2^{\mu\nu} = I^{\mu\nu}$. For the boost invariant symmetric solution the only nonzero components of the EMT are $T^{\chi\chi}$, $T^{ab} = \delta^{ab} T_T$ and $T^{00} = \tau^2 T^{\chi\chi} + 2T_T$. The only nontrivial Christoffel symbols are $\Gamma_{\chi\chi}^\tau = \tau$ and $\Gamma_{\tau\chi}^\chi = \tau^{-1}$. The conservation law implies

$$\frac{1}{2\tau} \frac{d}{d\tau} (\tau^2 T^{00}) = T_T \quad (13)$$

The nontrivial third moments are $A^{0\chi\chi}$, $A^{0ab} = \delta^{ab} A_T$, $A^{000} = \tau^2 A^{0\chi\chi} + 2A_T$ and their permutations. We now have

$$\begin{aligned} A_T &= \frac{\tau}{2} \int \frac{dp^\chi d^2 \mathbf{p}_\perp}{(2\pi)^3} p_\perp^2 f_0 [\tau^2 p^\chi, \mathbf{p}_\perp] \\ &= \frac{1}{2\tau} \int \frac{dP d^2 \mathbf{p}_\perp}{(2\pi)^3} p_\perp^2 f_0 [P, \mathbf{p}_\perp] = \frac{1}{\tau} A_{T0} \end{aligned} \quad (14)$$

$$\begin{aligned} A^{0\chi\chi} &= \tau \int \frac{dp^\chi d^2 \mathbf{p}_\perp}{(2\pi)^3} p^{\chi^2} f_0 [\tau^2 p^\chi, \mathbf{p}_\perp] \\ &= \frac{1}{\tau^5} \int \frac{dP d^2 \mathbf{p}_\perp}{(2\pi)^3} P^2 f_0 [P, \mathbf{p}_\perp] = \frac{1}{\tau^5} A_0^{0\chi\chi} \end{aligned} \quad (15)$$

These results can also be derived from the conservation law

$$A^{\mu\nu\rho}_{;\mu} = \frac{1}{\tau} \frac{d}{d\tau} (\tau A^{0\nu\rho}) + \Gamma_{\mu\lambda}^\nu A^{\mu\lambda\rho} + \Gamma_{\mu\lambda}^\rho A^{\mu\nu\lambda} = 0 \quad (16)$$

which follows from $I_2^{\mu\nu} = 0$ in eq. (11) for free streaming. $A_T \approx \tau^{-1}$ is evident. Setting $\nu = \rho = 1$ we get

$$\frac{1}{\tau} \frac{d}{d\tau} (\tau A^{0\chi\chi}) + \frac{4}{\tau} A^{0\chi\chi} = 0 \quad (17)$$

As expected. Finally, $\nu = \rho = 0$ leads to

$$\frac{1}{\tau} \frac{d}{d\tau} (\tau A^{000}) + 2\tau A^{0\chi\chi} = 0 \quad (18)$$

B. A restricted solution

As an example, let us consider the particular case where the 1PDF is a function only of the variable

$$\Xi^2 = C_0 p_\perp^2 + C_1 \tau^4 p^{\chi^2} \quad (19)$$

where the C_i are constants. We wish to compute momenta of the distribution function, which are of the form

$$\langle T_n \rangle = 2\tau \int_0^\infty dp^\chi \int \frac{d^2 \mathbf{p}_\perp}{(2\pi)^3 \sqrt{\tau^2 p^{\chi^2} + p_\perp^2}} T_n f_0 [C_1 \tau^4 p^{\chi^2} + C_0 p_\perp^2] \quad (20)$$

It is therefore convenient to introduce new variables

$$\begin{aligned} \frac{\phi}{C_0} &= c\tau^4 p^{\chi^2} + p_\perp^2 \\ \psi &= \tau^2 p^{\chi^2} + p_\perp^2 \end{aligned} \quad (21)$$

$c = C_1/C_0$. Then

$$\begin{aligned} p^{\chi^2} &= \frac{\frac{\phi}{C_0} - \psi}{\tau^2 (c\tau^2 - 1)} \\ p_\perp^2 &= \frac{c\tau^2 \psi - \frac{\phi}{C_0}}{(c\tau^2 - 1)} \end{aligned} \quad (22)$$

so

$$dp^\chi d^2 \mathbf{p}_\perp = \frac{\pi}{2p^\chi} dp^{\chi^2} d^2 \mathbf{p}_\perp = \frac{\pi}{2C_0 \tau \sqrt{c\tau^2 - 1}} \frac{d\phi d\psi}{\sqrt{\frac{\phi}{C_0} - \psi}} \quad (23)$$

Positivity of p^{χ^2} and p_\perp^2 implies

$$\frac{\phi}{C_0} \geq \psi \geq \frac{\phi}{C_1 \tau^2} \quad (24)$$

To compute T^{00} we choose $T_n = \psi$. We get

$$T^{00} = \frac{1}{8\pi^2 C_0 \sqrt{c\tau^2 - 1}} \int d\phi f_0[\phi] \int_{\phi/C_1 \tau^2}^{\phi/C_0} d\psi \frac{\sqrt{\psi}}{\sqrt{\frac{\phi}{C_0} - \psi}} \quad (25)$$

Leading to

$$T^{00} = \frac{J_1}{8\pi^2 C_0^2} \left[\frac{1}{c\tau^2} + \frac{\cos^{-1} \left[\frac{1}{\sqrt{c\tau}} \right]}{\sqrt{c\tau^2 - 1}} \right] \quad (26)$$

where

$$J_a = \int d\phi \phi^a f_0[\phi] \quad (27)$$

To compute T_T we set $T_n = p_\perp^2/2$

$$T_T = \frac{J_1}{16\pi^2 C_0^2 (c\tau^2 - 1)^{3/2}} \int_{1/c\tau^2}^1 dt \frac{c\tau^2 t - 1}{\sqrt{t}\sqrt{1-t}} \quad (28)$$

We may also get T_T from the conservation law (we write $s = c\tau^2$)

$$T_T = \frac{d}{ds} (sT^{00}) = \frac{J_1}{16\pi^2 C_0^2 (c\tau^2 - 1)} \left[1 + \frac{(c\tau^2 - 2)}{\sqrt{c\tau^2 - 1}} \cos^{-1} \left[\frac{1}{\sqrt{c\tau}} \right] \right] \quad (29)$$

To compute $A^{0\chi\chi}$ we set $T_n = \psi^{1/2} p_{\chi^2}$

$$A^{0\chi\chi} = \frac{1}{8\pi^2 C_0 \tau^2 (c\tau^2 - 1)^{3/2}} \int d\phi f_0[\phi] \int_{\phi/C_1 \tau^2}^{\phi/C_0} d\psi \sqrt{\frac{\phi}{C_0} - \psi} = \frac{J_{3/2}}{12\pi^2 C_0 C_1^{3/2} \tau^5} \quad (30)$$

To compute A_T we set $T_n = \psi^{1/2} p_{\perp}^2 / 2$

$$A_T = \frac{1}{16\pi^2 C_0 (c\tau^2 - 1)^{3/2}} \int d\phi f_0[\phi] \int_{\phi/C_1 \tau^2}^{\phi/C_0} d\psi \frac{c\tau^2 \psi - \frac{\phi}{C_0}}{\sqrt{\frac{\phi}{C_0} - \psi}} = \frac{J_{3/2}}{12\pi^2 C_0^2 C_1^{1/2} \tau} \quad (31)$$

III. A GENERIC (DTT-INSPIRED) NON LINEAR HYDRO

We wish to see if it is possible to cast hydrodynamics into something resembling a DTT framework, with the goal to provide a fully nonlinear model. For simplicity we consider a conformal, neutral fluid.

The degrees of freedom are a four-vector β_μ and a traceless, transverse symmetric tensor $\zeta_{\mu\nu}$. The equations of motion are of divergence type

$$\begin{aligned} T_{;\mu}^{\mu\nu} &= 0 \\ A_{;\mu}^{\mu\nu\rho} &= I^{\nu\rho} \end{aligned} \quad (32)$$

$T^{\mu\nu}$ is traceless, $A^{\mu\nu\rho}$ is totally symmetric and traceless on any two indexes. We expect they are derivable from a potential

$$\begin{aligned} T^{\mu\nu} &= \frac{\partial \Phi^\mu}{\partial \beta_\nu} \\ A^{\mu\nu\rho} &= \frac{\partial \Phi^\mu}{\partial \zeta_{\nu\rho}} \end{aligned} \quad (33)$$

The symmetry of the EMT implies that Φ^μ itself is a gradient

$$\Phi^\nu = \frac{\partial \Phi}{\partial \beta_\nu} \quad (34)$$

Since Φ is a scalar, it can only depend on other scalars. Of course we must include scalars which are zero on-shell, but may contribute to variations. On the other hand, only the first four powers of $\zeta_{\mu\nu}$ may be regarded as independent, because there are no more than four independent invariants. We introduce variables

$$\begin{aligned} x &= -\beta_\mu \beta^\mu \\ y^j &= (\zeta^j)_\mu^\mu \\ z^j &= \beta_\mu (\zeta^j)_\nu^\mu \beta^\nu \end{aligned} \quad (35)$$

$1 \leq j \leq 4$. We get

$$\Phi^\mu = -2\beta^\mu \frac{\partial \Phi}{\partial x} + 2 \sum_{j=1}^4 (\zeta^j)^\mu_\nu \beta^\nu \frac{\partial \Phi}{\partial z^j} \quad (36)$$

Therefore

$$T^{\mu\nu} = -2g^{\mu\nu} \frac{\partial \Phi}{\partial x} + 4\beta^\mu \beta^\nu \frac{\partial^2 \Phi}{\partial x^2} + 2 \sum_{j=1}^4 (\zeta^j)^{\mu\nu} \frac{\partial \Phi}{\partial z^j} \quad (37)$$

Plus terms which vanish on shell, where $\zeta^{\mu\nu}$ is transverse, and

$$\begin{aligned} A^{\mu\nu\rho} = & -2\beta^\mu \sum_{j=1}^4 j (\zeta^{j-1})^{\nu\rho} \frac{\partial^2 \Phi}{\partial y^j \partial x} - 2\beta^\mu \beta^\nu \beta^\rho \frac{\partial^2 \Phi}{\partial z^1 \partial x} \\ & + \sum_{j=1}^4 \left[(\zeta^{j-1})^{\mu\nu} \beta^\rho + (\zeta^{j-1})^{\mu\rho} \beta^\nu \right] \frac{\partial \Phi}{\partial z^j} \end{aligned} \quad (38)$$

plus terms which vanish on shell. Tracelessness of $T^{\mu\nu}$ implies

$$0 = -8 \frac{\partial \Phi}{\partial x} - 4x \frac{\partial^2 \Phi}{\partial x^2} + 2 \sum_{j=1}^4 y^j \frac{\partial \Phi}{\partial z^j} \quad (39)$$

The symmetry of $A^{\mu\nu\rho}$ suggests that, at least on shell

$$2j \frac{\partial^2 \Phi}{\partial y^j \partial x} = - \frac{\partial \Phi}{\partial z^j} \quad (40)$$

The solution to these two equations is

$$\frac{\partial \Phi}{\partial x} = \frac{-1}{6x^2} F[r^j] \quad (41)$$

where $r^j = x^{-j} y^j$ and F is an arbitrary function. Write

$$\frac{\partial F}{\partial y^j} = x^{-j} \frac{\partial F}{\partial r^j} \quad (42)$$

and

$$g^{\mu\nu} = \Delta^{\mu\nu} - \frac{1}{x} \beta^\mu \beta^\nu \quad (43)$$

Then

$$T^{\mu\nu} = \left[\frac{1}{x} \beta^\mu \beta^\nu + \frac{1}{3} \Delta^{\mu\nu} \right] \frac{1}{x^2} \left\{ F + \frac{2}{3} \sum_{j=1}^4 j r^j \frac{\partial F}{\partial r^j} \right\} + \frac{2}{3x^2} \sum_{j=1}^4 j \left[x^{-j} (\zeta^j)^{\mu\nu} - \frac{1}{3} r^j \Delta^{\mu\nu} \right] \frac{\partial F}{\partial r^j} \quad (44)$$

and

$$A^{\mu\nu\rho} = \frac{-2}{3}\beta^\mu\beta^\nu\beta^\rho\frac{\partial}{\partial x}\frac{F_{,1}}{x^3} + \frac{1}{3x^2}\sum_{j=1}^4 jx^{-j}\left[(\zeta^{j-1})^{\nu\rho}\beta^\mu + (\zeta^{j-1})^{\mu\nu}\beta^\rho + (\zeta^{j-1})^{\mu\rho}\beta^\nu\right]\frac{\partial F}{\partial r^j} \quad (45)$$

Contracting ν and ρ we get

$$0 = x\frac{\partial}{\partial x}\frac{\partial F}{\partial r^1} + \frac{1}{2}\sum_{j=2}^4 jr^{j-1}\frac{\partial F}{\partial r^j} \quad (46)$$

At this point, we parametrize $\beta^\mu = u^\mu/T$, $\zeta^{\mu\nu} = \xi^{\mu\nu}/T^2$ and $x = 1/T^2$ to get $r^j = (\xi^j)^\lambda_\lambda$

$$\begin{aligned} T^{\mu\nu} &= T^4 \left\{ \left[u^\mu u^\nu + \frac{1}{3}\Delta^{\mu\nu} \right] \left\{ F + \frac{2}{3}\sum_{j=1}^4 jr^j \frac{\partial F}{\partial r^j} \right\} + \frac{2}{3}\sum_{j=1}^4 j \left[(\xi^j)^{\mu\nu} - \frac{1}{3}(\xi^j)^\lambda_\lambda \Delta^{\mu\nu} \right] \frac{\partial F}{\partial r^j} \right\} \\ A^{\mu\nu\rho} &= T^5 \left\{ \left[u^\mu u^\nu u^\rho + \frac{1}{3}[\Delta^{\nu\rho}u^\mu + \Delta^{\mu\nu}u^\rho + \Delta^{\mu\rho}u^\nu] \right] F_{,1} \right. \\ &\quad \left. + \frac{1}{3}\sum_{j=2}^4 j \left[(\xi^{j-1})^{\nu\rho}u^\mu + (\xi^{j-1})^{\mu\nu}u^\rho + (\xi^{j-1})^{\mu\rho}u^\nu + u^\mu u^\nu u^\rho (\xi^{j-1})^\lambda_\lambda \right] \frac{\partial F}{\partial r^j} \right\} \end{aligned} \quad (47)$$

On shell, ξ is both traceless and transverse. Therefore, it can be written in some frame as

$$\xi^\mu_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \xi_+ + \xi_- & 0 & 0 \\ 0 & 0 & \xi_+ - \xi_- & 0 \\ 0 & 0 & 0 & -2\xi_+ \end{pmatrix} \quad (48)$$

This implies

$$(\xi^3)^\mu_\nu = \frac{1}{3}(\xi^3)^\rho_\rho \Delta^\mu_\nu + \frac{1}{2}(\xi^2)^\rho_\rho \xi^\mu_\nu \quad (49)$$

$$(\xi^4)^\mu_\nu = \frac{1}{3}(\xi^3)^\rho_\rho \xi^\mu_\nu + \frac{1}{2}(\xi^2)^\rho_\rho (\xi^2)^\mu_\nu \quad (50)$$

Observe that $(\xi^2)^\mu_\mu = 2(\xi_-^2 + 3\xi_+^2)$, $(\xi^3)^\mu_\mu = 6\xi_+(\xi_-^2 - \xi_+^2)$ and $(\xi^4)^\mu_\mu = 2(\xi_-^2 + 3\xi_+^2)^2$. We now get

$$\begin{aligned} T^{\mu\nu} &= T^4 \left\{ \left[u^\mu u^\nu + \frac{1}{3}\Delta^{\mu\nu} \right] \left[F + \frac{2}{3}\sum_{j=1}^4 jr^j \frac{\partial F}{\partial r^j} \right] \right. \\ &\quad \left. + \xi^{\mu\nu} \left[\frac{2}{3}\frac{\partial F}{\partial r^1} + (\xi^2)^\rho_\rho \frac{\partial F}{\partial r^3} + \frac{8}{9}(\xi^3)^\rho_\rho \frac{\partial F}{\partial r^4} \right] + \frac{4}{3}(\tilde{\xi}^2)^{\mu\nu} \left[\frac{\partial F}{\partial r^2} + (\xi^2)^\rho_\rho \frac{\partial F}{\partial r^4} \right] \right\} \\ A^{\mu\nu\rho} &= T^5 \left\{ \left[u^\mu u^\nu u^\rho + \frac{1}{3}[\Delta^{\nu\rho}u^\mu + \Delta^{\mu\nu}u^\rho + \Delta^{\mu\rho}u^\nu] \right] \left[F_{,1} + (\xi^2)^\lambda_\lambda \frac{\partial F}{\partial r^3} + \frac{4}{3}(\xi^3)^\lambda_\lambda \frac{\partial F}{\partial r^4} \right] \right. \\ &\quad \left. + \frac{2}{3}[\xi^{\nu\rho}u^\mu + \xi^{\mu\nu}u^\rho + \xi^{\mu\rho}u^\nu] \left[\frac{\partial F}{\partial r^2} + (\xi^2)^\lambda_\lambda \frac{\partial F}{\partial r^4} \right] \right. \\ &\quad \left. + \left[(\tilde{\xi}^2)^{\nu\rho}u^\mu + (\tilde{\xi}^2)^{\mu\nu}u^\rho + (\tilde{\xi}^2)^{\mu\rho}u^\nu \right] \frac{\partial F}{\partial r^3} \right\} \end{aligned} \quad (51)$$

$$(\tilde{\xi}^2)^{\mu\nu} = (\xi^2)^{\mu\nu} - \frac{1}{3}(\xi^2)^\lambda_\lambda \Delta^{\mu\nu} \quad (52)$$

Equations (51) and (52) are the most general expressions for the EMT and NEC for theories derived from a potential. They are the most important result of this paper, because they display in full the inner relations between the transport functions appearing in one and the other. Of course, for boost invariant, axisymmetric flows, these equations take a much simpler form, which we derive presently.

A. Restricted hydrodynamic solutions

In this subsection we shall derive the simplified form of equations (51) and (52) valid for boost invariant, axisymmetric flows.

If the solution is axisymmetric, then $\xi_- = 0$ and the only nontrivial parameter is $\xi = \xi_+$. We have $(\tilde{\xi}^2)^{\mu\nu} = -\xi\xi^{\mu\nu}$ and equations (51) and (52) become

$$\begin{aligned} T^{\mu\nu} &= T^4 \left\{ \left[u^\mu u^\nu + \frac{1}{3} \Delta^{\mu\nu} \right] \left[F + \frac{2}{3} \sum_{j=1}^4 j r^j \frac{\partial F}{\partial r^j} \right] \right. \\ &\quad \left. + \xi^{\mu\nu} \left[\frac{2}{3} \frac{\partial F}{\partial r^1} - \frac{4}{3} \xi \frac{\partial F}{\partial r^2} + 6\xi^2 \frac{\partial F}{\partial r^3} - \frac{40}{3} \xi^3 \frac{\partial F}{\partial r^4} \right] \right\} \\ A^{\mu\nu\rho} &= T^5 \left\{ \left[u^\mu u^\nu u^\rho + \frac{1}{3} [\Delta^{\nu\rho} u^\mu + \Delta^{\mu\nu} u^\rho + \Delta^{\mu\rho} u^\nu] \right] \left[F_{,1} + 6\xi^2 \frac{\partial F}{\partial r^3} - 8\xi^3 \frac{\partial F}{\partial r^4} \right] \right. \\ &\quad \left. + \frac{2}{3} [\xi^{\nu\rho} u^\mu + \xi^{\mu\nu} u^\rho + \xi^{\mu\rho} u^\nu] \left[\frac{\partial F}{\partial r^2} - \frac{3}{2} \xi \frac{\partial F}{\partial r^3} + 6\xi^2 \frac{\partial F}{\partial r^4} \right] \right\} \end{aligned} \quad (53)$$

In Milne coordinates $u^\mu = \delta_0^\mu$. The nontrivial components of the nonequilibrium current are $A^{0\chi\chi}$ and $A^{0ab} = \delta^{ab} A_T$

$$A_T = \frac{T^5}{3} \{ F_{,1} + 2\xi F_{,2} + 3\xi^2 F_{,3} + 4\xi^3 F_{,4} \} \quad (54)$$

$$A^{0\chi\chi} = \frac{T^5}{3\tau^2} \{ F_{,1} - 4\xi F_{,2} + 12\xi^2 F_{,3} - 32\xi^3 F_{,4} \} \quad (55)$$

The energy momentum tensor

$$T^{00} = T^4 \left[F + \frac{2}{3} \sum_{j=1}^4 j r^j \frac{\partial F}{\partial r^j} \right] = T^4 [F + 4(2\xi^2 F_{,2} - 3\xi^3 F_{,3} + 12\xi^4 F_{,4})] \quad (56)$$

$$T_T = \frac{1}{3} T^{00} + \frac{2}{3} \xi T^4 [F_{,1} - 2\xi F_{,2} + 9\xi^2 F_{,3} - 20\xi^3 F_{,4}] \quad (57)$$

IV. MATCHING HYDRODYNAMICS TO KINETIC THEORY

We now compare these formulae for the conserved currents obtained from kinetic theory (equations (26), (29), (30) and (31)) with the ones obtained in the hydrodynamic framework, namely equations (56), (57), (55) and (54), respectively.

The first point to be made is that the left hand sides of eqs. (56), (57), (55) and (54) are not linearly independent, rather they imply the relationship

$$T_T = \frac{1}{3} T^{00} + \frac{2\xi}{3T} [A_T + 2\tau^2 A^{0\chi\chi}] \quad (58)$$

Therefore a successful match requires

$$\frac{\xi}{T} = \frac{3 [T_T - \frac{1}{3} T^{00}]}{2 [A_T + 2\tau^2 A^{0\chi\chi}]} = \frac{2J_1 C_0^{1/2}}{15J_{3/2}} H(z) \quad (59)$$

where $z = c\tau^2$ and

$$H(z) = \frac{45}{16} \frac{z^{1/2}}{z-1} \left[1 + \frac{z(z-4)}{z+2} \frac{\cos^{-1} \left[\frac{1}{z^{1/2}} \right]}{\sqrt{z-1}} \right] \quad (60)$$

Matching the NEC components yields

$$\begin{aligned} \xi^5 \{F_{,1} + 2\xi F_{,2} + 3\xi^2 F_{,3} + 4\xi^3 F_{,4}\} &= \frac{J_{3/2}}{4\pi^2 c^{1/2} \tau} \left[\frac{2J_1}{15J_{3/2}} H(z) \right]^5 \\ \xi^5 \{F_{,1} - 4\xi F_{,2} + 12\xi^2 F_{,3} - 32\xi^3 F_{,4}\} &= \frac{J_{3/2}}{4\pi^2 c^{3/2} \tau^3} \left[\frac{2J_1}{15J_{3/2}} H(z) \right]^5 \end{aligned} \quad (61)$$

Therefore

$$\xi^5 \{6\xi F_{,2} - 9\xi^2 F_{,3} + 36\xi^3 F_{,4}\} = \frac{J_{3/2}}{4\pi^2 c^{3/2} \tau^3} \left[\frac{2J_1}{15J_{3/2}} H(z) \right]^5 (z-1) \quad (62)$$

Matching T^{00} yields

$$\xi^4 [F + 4(2\xi^2 F_{,2} - 3\xi^3 F_{,3} + 12\xi^4 F_{,4})] = \frac{J_1}{8\pi^2} \left[\frac{1}{z} + \frac{\cos^{-1} \left[\frac{1}{z^{1/2}} \right]}{\sqrt{z-1}} \right] \left[\frac{2J_1}{15J_{3/2}} H(z) \right]^4 \quad (63)$$

We postulate a solution

$$F = \sum_{a_2, a_3, a_4} \frac{f_{a_2, a_3, a_4}}{6^{a_2} (-6)^{a_3} 18^{a_4}} (r^2)^{a_2} (r^3)^{a_3} (r^4)^{a_4} = \sum_{a_1, a_2, a_3} f_{a_2, a_3, a_4} \xi^{2a_2+3a_3+4a_4} \quad (64)$$

Then

$$\begin{aligned} \xi^2 F_{,2} &= \sum_{a_2, a_3, a_4} a_2 \frac{f_{a_2, a_3, a_4}}{6^{a_2} (-6)^{a_3} 18^{a_4}} \xi^2 (r^2)^{a_2-1} (r^3)^{a_3} (r^4)^{a_4} = \frac{1}{6} \sum_{a_1, a_2, a_3} a_2 f_{a_2, a_3, a_4} \xi^{2a_2+3a_3+4a_4} \\ \xi^3 F_{,3} &= \sum_{a_2, a_3, a_4} a_3 \frac{f_{a_2, a_3, a_4}}{6^{a_2} (-6)^{a_3} 18^{a_4}} \xi^3 (r^2)^{a_2} (r^3)^{a_3-1} (r^4)^{a_4} = \frac{-1}{6} \sum_{a_1, a_2, a_3} a_3 f_{a_2, a_3, a_4} \xi^{2a_2+3a_3+4a_4} \\ \xi^4 F_{,4} &= \sum_{a_2, a_3, a_4} a_4 \frac{f_{a_2, a_3, a_4}}{6^{a_2} (-6)^{a_3} 18^{a_4}} \xi^4 (r^2)^{a_2} (r^3)^{a_3} (r^4)^{a_4-1} = \frac{1}{18} \sum_{a_1, a_2, a_3} a_4 f_{a_2, a_3, a_4} \xi^{2a_2+3a_3+4a_4} \end{aligned} \quad (65)$$

But then

$$2\xi^2 F_{,2} - 3\xi^3 F_{,3} + 12\xi^4 F_{,4} = \frac{1}{6} \xi \frac{dF}{d\xi} \quad (66)$$

Therefore

$$\xi^5 \frac{dF}{d\xi} = \frac{J_{3/2}}{2\pi^2 z^{3/2}} \left[\frac{2J_1}{15J_{3/2}} H(z) \right]^5 (z-1) = \frac{J_1}{15\pi^2} \left[\frac{2J_1}{15J_{3/2}} \right]^4 H_1(z) \quad (67)$$

and

$$\xi^4 F = \frac{J_1}{4\pi^2} \left[\frac{2J_1}{15J_{3/2}} H(z) \right]^4 \left\{ \frac{1}{2z} + \frac{\cos^{-1} \left[\frac{1}{z^{1/2}} \right]}{2\sqrt{z-1}} - \frac{8}{45z^{3/2}} H(z)(z-1) \right\} = \frac{J_1}{4\pi^2} \left[\frac{2J_1}{15J_{3/2}} \right]^4 H_2(z) \quad (68)$$

We get rid of the constants by defining

$$\begin{aligned} F &= \frac{J_1}{4\pi^2} f \\ \xi &= \sqrt{5} \frac{2J_1}{5J_{3/2}} x \end{aligned} \quad (69)$$

We get

$$\begin{aligned} (3\sqrt{5}x)^4 f &= H_2(z) \\ (3\sqrt{5})^4 x^5 \frac{df}{dx} &= \frac{4}{15} H_1(z) \end{aligned} \quad (70)$$

But then

$$(3\sqrt{5})^4 x \frac{d}{dx} x^4 f = \frac{dH_2}{dz} x \frac{dz}{dx} = \frac{4}{15} H_1(z) + 4H_2(z) \quad (71)$$

or else

$$\frac{dx}{dz} = \frac{1}{4} \frac{x}{H_2 + \frac{1}{15} H_1} \frac{dH_2}{dz} \quad (72)$$

Although this equation may be solved numerically, the resulting theory is likely to be too complex to be of any use. We shall explore a simple approximation instead.

When $z \rightarrow 1$, we find

$$\begin{aligned} H(z) &= (t-1) - \frac{5}{42} (t-1)^2 \\ H_1(z) &= (t-1)^6 \left[1 - \frac{44}{21} (t-1) \right] \\ H_2(z) &= (t-1)^4 \left[1 - \frac{8}{7} (t-1) + \frac{1026}{2205} (t-1)^2 \right] \\ H_2 + \frac{1}{15} H_1 &= (t-1)^4 \left[1 - \frac{8}{7} (t-1) + \frac{1213}{2205} (t-1)^2 \right] \end{aligned} \quad (73)$$

Since $H_1 \ll H_2$ the simplest approximation is $x = (3\sqrt{5})^{-1} H_2^{1/4}$, $f = 1$. The next approximation is obtained by writing

$$\frac{dx}{dz} = \frac{1}{4} \frac{x}{H_2} \frac{dH_2}{dz} \left[1 - \frac{1}{15} \frac{H_1}{H_2} \right] \approx \frac{1}{4} \frac{x}{H_2} \frac{dH_2}{dz} \left[1 - \frac{1}{15} H_2^{1/2} \right] \quad (74)$$

so, if $x \approx (3\sqrt{5})^{-1} (H_2^{1/4} + bH_2^{3/4})$ we get

$$\frac{1}{4} \frac{1}{H_2} \frac{dH_2}{dz} [H_2^{1/4} + 3bH_2^{3/4}] = \frac{1}{4} \frac{1}{H_2} \frac{dH_2}{dz} \left[H_2^{1/4} + \left(b - \frac{1}{15} \right) H_2^{3/4} \right] \quad (75)$$

Therefore $b = -1/30$. Now $x^4 \approx (3\sqrt{5})^{-4} H_2 \left(1 - 2H_2^{1/2}/15\right)$, and $f \approx 1 + 2H_2^{1/2}/15$, leading to

$$F = \frac{J_1}{4\pi^2} [1 + 6x^2] = \frac{J_1}{4\pi^2} \left\{ 1 + \frac{15}{2} \frac{J_{3/2}^2}{J_1^2} \frac{r^2}{6} \right\} \quad (76)$$

$$F_{,2} = \frac{5}{4} \frac{J_{3/2}^2}{4\pi^2 J_1} \quad (77)$$

$F_{,3} = F_{,4} = 0$. Once F , $F_{,2}$, $F_{,3}$ and $F_{,4}$ are known, $F_{,1}$ may be found from the equations above. It is also possible to choose $F_{,1}$ to enforce the compatibility of all three nontrivial conservation laws. Let us write

$$\begin{aligned} A_T &= \frac{T^5}{3} F_{AT}(\xi) \\ A^{0\chi\chi} &= \frac{T^5}{3\tau^2} F_{A\chi}(\xi) \\ T^{00} &= T^4 F_{T0}(\xi) \\ T_T &= T^4 F_{TT}(\xi) \end{aligned} \quad (78)$$

The conservation laws for the NEC imply

$$\begin{aligned} 1 + 5 \frac{\tau}{T} \frac{dT}{d\tau} + \frac{F'_{AT}}{F_{AT}} \tau \frac{d\xi}{d\tau} &= 0 \\ 3 + 5 \frac{\tau}{T} \frac{dT}{d\tau} + \frac{F'_{A\chi}}{F_{A\chi}} \tau \frac{d\xi}{d\tau} &= 0 \end{aligned} \quad (79)$$

so

$$\begin{aligned} \tau \frac{d\xi}{d\tau} &= 2 \left[\frac{F'_{AT}}{F_{AT}} - \frac{F'_{A\chi}}{F_{A\chi}} \right]^{-1} \\ \frac{\tau}{T} \frac{dT}{d\tau} &= \frac{-1}{5} \left[\frac{F'_{AT}}{F_{AT}} - \frac{F'_{A\chi}}{F_{A\chi}} \right]^{-1} \left[3 \frac{F'_{AT}}{F_{AT}} - \frac{F'_{A\chi}}{F_{A\chi}} \right] \end{aligned} \quad (80)$$

Write EMT conservation as

$$\frac{\tau}{2} \frac{d}{d\tau} \ln [\tau^2 T^{00}] = \frac{T_T}{T^{00}} \quad (81)$$

This becomes

$$\frac{-2}{5} F_{T0} [3F_{A\chi} F'_{AT} - F_{AT} F'_{A\chi}] + F_{AT} F_{A\chi} F'_{T0} = [F_{TT} - F_{T0}] [F_{A\chi} F'_{AT} - F_{AT} F'_{A\chi}] \quad (82)$$

Define a new unknown

$$F_{,1} = \frac{J_{3/2}}{12\pi^2} f_1 \quad (83)$$

Then

$$\begin{aligned} F_{AT} &= F_{,1} + 2\xi F_{,2} = \frac{J_{3/2}}{12\pi^2} [f_1 + 3\sqrt{5}x] \\ F_{A\chi} &= F_{,1} - 4\xi F_{,2} = \frac{J_{3/2}}{12\pi^2} [f_1 - 6\sqrt{5}x] \\ F_{T0} &= F + 8\xi^2 F_{,2} = \frac{J_1}{4\pi^2} [1 + 14x^2] \\ F_{TT} &= \frac{1}{3} F_{T0} + \frac{2}{3} \xi [F_{,1} - 2\xi F_{,2}] = \frac{J_1}{12\pi^2} \left[1 + 2x^2 + \frac{4}{\sqrt{5}} x f_1 \right] \end{aligned} \quad (84)$$

We obtain an equation for f_1

$$f_1' = \frac{A + Bf_1 + Cf_1^2}{D + Ef_1} \quad (85)$$

$$\begin{aligned} A &= 378x^3 - 18x \\ B &= 12\sqrt{5}x^2 \\ C &= -4x \\ D &= \frac{3}{5}\sqrt{5}x[1 - x^2] \\ E &= 3x - \frac{1}{5}[1 + 14x^2] \end{aligned} \quad (86)$$

with the boundary condition $f_1(0) = 1$ as follows from the perturbative analysis above. Since it is seen that $f_1' = O(x)$ when $x \rightarrow 0$, $f_1 = 1$ is an admissible approximation within the sought accuracy.

A. Dynamics from the truncated potential

We shall conclude this paper by comparing the dynamics as obtained from the truncated theory and the exact solution. The truncated theory yields the currents

$$\begin{aligned} A_T &= \frac{J_{3/2}}{12\pi^2} \frac{T^5}{3} [1 + 3\sqrt{5}x] \\ A^{0\chi\chi} &= \frac{J_{3/2}}{12\pi^2} \frac{T^5}{3\tau^2} [1 - 6\sqrt{5}x] \\ T_T &= \frac{J_1}{12\pi^2} T^4 \left[1 + 2x^2 + \frac{4}{\sqrt{5}}x \right] \end{aligned} \quad (87)$$

We seek trajectories which begin at $x = 0$ at some time τ_0 , when the temperature is T_0 . Let $t = \tau/\tau_0$. Then the dynamics is determined by the conservation laws for $A_T \approx t^{-1}$ and $A^{0\chi\chi} \approx t^{-5}$. Taking the ratio of these two we get

$$\frac{1 + 3\sqrt{5}x}{1 - 6\sqrt{5}x} = t^2 \quad (88)$$

which is easily inverted to

$$x = \frac{\sqrt{5}}{30} \frac{t^2 - 1}{t^2 + (1/2)} \quad (89)$$

x saturates at a finite value. The point is however that it is non decreasing, meaning that the theory has no isotropization mechanism built in.

Given x , we find T as

$$\frac{T}{T_0} = \left[\frac{1}{t} \frac{1}{1 + 3\sqrt{5}x} \right]^{1/5} = \left[\frac{2}{3t^3} \left(t^2 + \frac{1}{2} \right) \right]^{1/5} \quad (90)$$

T_T is found by simple substitution; we display the result in fig. (1)

Because we are approximating $f_1 = 1$ rather than solving eq. (85), if we derived T^{00} directly from the potential we would not be enforcing EMT conservation exactly. It is more accurate to derive T_T , as we have done, and then find T^{00} from the conservation law, namely

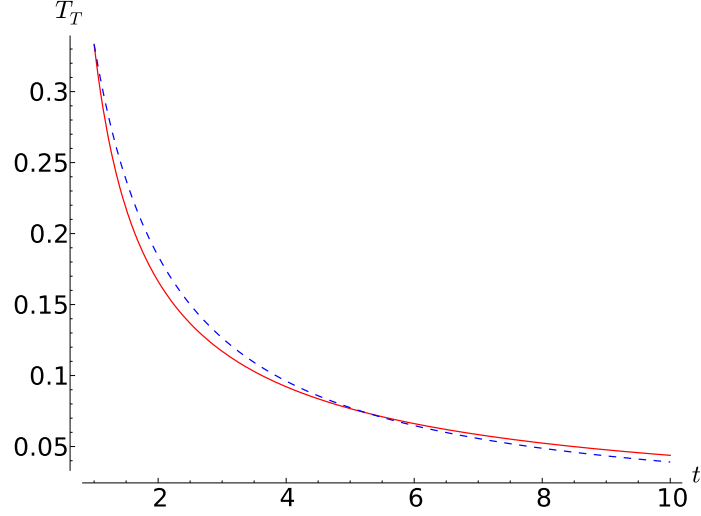


Figure 1: (Color online) T_T , scaled by $J_1 T_0^4 / 4\pi^2$, as obtained in the truncated theory (full line, red) and in the exact solution (dashed, blue), as a function of $t = \tau / \tau_0$.

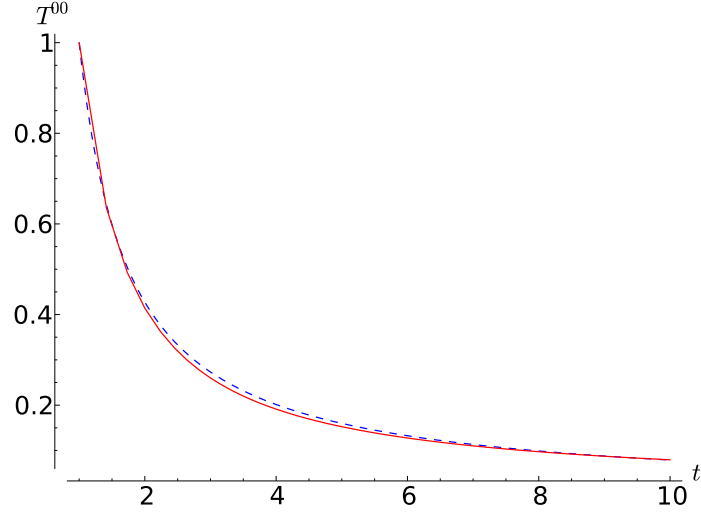


Figure 2: (Color online) T^{00} , scaled by $J_1 T_0^4 / 4\pi^2$, as obtained in the truncated theory from eq. (91) (full line, red) and in the exact solution (dashed, blue), as a function of $t = \tau / \tau_0$.

$$T^{00} = \frac{J_1}{4\pi^2} \frac{1}{t^2} + \frac{1}{t^2} \int_1^t dt' 2t' T_T(t') \quad (91)$$

We display the result in fig. (2)

Although the result is remarkably accurate, we see that for times $t \geq 5$ our approximation overestimates T_T . For this reason the pressure anisotropy, defined as

$$\frac{P_L}{P_T} = \frac{T^{00} - 2T_T}{T_T} \quad (92)$$

eventually becomes negative (see fig. (3)). This event marks the limit of validity of our approximation. The point is, of course, that up to this point the approximation captures the trend in the development of the pressure anisotropy.

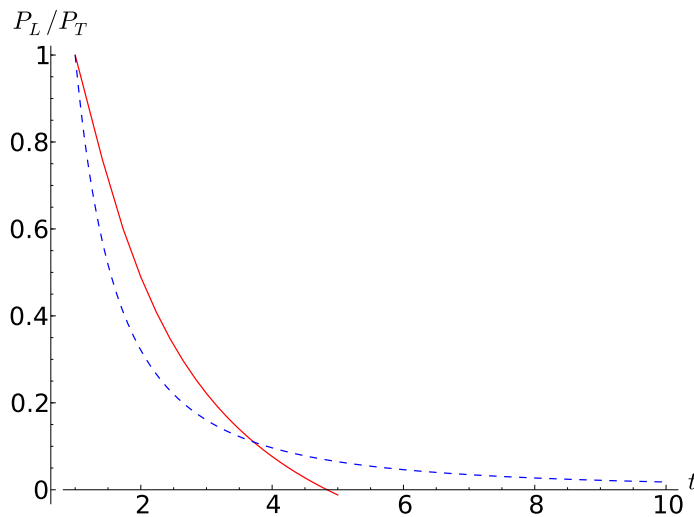


Figure 3: (Color online) Pressure anisotropy, defined in eq. (92), as obtained in the truncated theory (full line, red) and in the exact solution (dashed, blue), as a function of $t = \tau/\tau_0$.

V. FINAL REMARKS

The main contribution of this paper is that we display the most general nonlinear hydrodynamic theory based on two conserved currents, the EMT and the NEC, having the symmetries of the second and third moments of the 1PDF, and where these currents may be derived from a vector potential.

Formally, the existence of a potential is essential to give the approach some predictive power; without it, there are just too many possibilities. The derivability from a potential ensures that there must be relations between the transport functions that appear in the conservation equation for the EMT and in the NEC. Those relations are the real predictions of the theory.

The existence of a potential, on the other hand, follows naturally if there is a non equilibrium entropy current S^μ and the Second law (positivity of the entropy production $S^\mu_{;\mu}$) has to follow from the conservation laws at every single event. This means that we should be able to write a local relation

$$S^\mu_{;\mu} = -\beta_\nu T^{\mu\nu}_{;\mu} - \zeta_{\rho\sigma} A^{\mu\rho\sigma}_{;\mu} \quad (93)$$

and this in turn suggests

$$dS^\mu = -\beta_\nu dT^{\mu\nu} - \zeta_{\rho\sigma} dA^{\mu\rho\sigma} \quad (94)$$

whereby one would find the potential as a local Legendre transform of the entropy current. However, a general proof that hydrodynamics, as derived from kinetic theory, admits a potential, has remained elusive [65]. Of course, both ideal hydrodynamics and first order theories do admit a potential [70–73].

The proof that given any exact boost invariant, axisymmetric, free streaming solution there is some potential leading to currents matching the exact ones is just a tour de force to demonstrate the flexibility afforded by these theories. More interesting is the observation that theories with very simple potentials already capture the early time dynamics. The point is that these theories are fully nonlinear, and so provide a natural framework to further analyze strong phenomena such as turbulence and instabilities. For these purposes, of course, it will be necessary to expand the present framework to account for a finite relaxation time and interaction with color fields [74, 75].

The three fields where to expect these simplified theories will prove useful are the nonlinear unfolding of plasma instabilities [22–29, 52–55], strong shocks [31–38] and turbulence [39–42]. We expect to report soon on progress in these directions.

Acknowledgments

It is a pleasure to acknowledge exchanges with J. Peralta-Ramos, P. Romatschke, L. Lindblom and M. Strickland. This work has been supported in part by ANPCyT, CONICET and University of Buenos Aires (Argentina).

-
- [1] BRAHMS Collaboration, Quark-gluon plasma and color glass condensate at RHIC? The perspective from the BRAHMS experiment, Nucl. Phys. A757, 1 (2005).
 - [2] PHOBOS Collaboration, The PHOBOS perspective on discoveries at RHIC, Nucl. Phys. A757, 28 (2005).
 - [3] STAR Collaboration, Experimental and theoretical challenges in the search for the quark-gluon plasma: The STAR Collaboration's critical assesment of the evidence from RHIC collisions, Nucl. Phys. A757, 102 (2005).
 - [4] PHENIX Collaboration, Formation of dense partonic matter in relativistic nucleus-nucleus collisions at RHIC: Experimental evaluation by the PHENIX collaboration, Nucl. Phys. A757, 184 (2005).
 - [5] R. Vogt, *Ultrarelativistic heavy-ion colisions* (Elsevier, Amsterdam, 2007)
 - [6] S. Sarkar, H. Satz and B. Sinha (Eds.) *The Physics of the Quark-Gluon Plasma* (Springer-Verlag, Berlin, 2010)
 - [7] W. Israel, Covariant fluid mechanics and thermodynamics: an introduction, in A. Anile and Y. Choquet - Bruhat (eds.), *Relativistic fluid dynamics* (Springer, New York, 1988).
 - [8] J.D. Bjorken, Highly relativistic nucleus-nucleus collisions: the central rapidity region, Phys. Rev. D **27**, 140 (1983).
 - [9] P. V. Ruuskanen, Transverse hydrodynamics with a first order phase transition in very high energy nuclear collisions, Acta Phys. Pol. B18, 551 (1986)
 - [10] D. Rischke, Fluid dynamics for relativistic nuclear collisions, Proceedings of the 11th Chris Engelbrecht Summer School in Theoretical Physics, Cape Town, Feb. 4 - 13, 1998
 - [11] E. Calzetta and B-L Hu, *Nonequilibrium quantum field theory* (Cambridge University Press, Cambridge (England), 2008)
 - [12] Tetsufumi Hirano, Naomi van der Kolk and Ante Bilandzic, Hydrodynamics and Flow, ArXiv 0808.2684
 - [13] P. Romatschke, New Developments in Relativistic Viscous Hydrodynamics, Int.J.Mod.Phys.E19:1-53,2010 (ArXiv 0902.3663)
 - [14] T. Hirano and Y. Nara, Dynamical modeling of high energy heavy ion collisions, arXiv:1203.4418
 - [15] C. Gale, S. Jeon and B. Schenke, Hydrodynamic Modeling of Heavy-Ion Collisions, Int. J. of Mod. Phys. A, Vol. 28, 1340011 (2013) (arXiv:1301.5893)
 - [16] P. Huovinen, Hydrodynamics at RHIC and LHC: What have we learned?, Int. J. of Mod. Phys. E22 (2013) 1330029 (arXiv:1311.1849)
 - [17] T. Hirano, Theory Summary, arXiv:1402.0913
 - [18] E. Calzetta, Real relativistic fluids in heavy ion collisions, arXiv:1310.0841
 - [19] M. Strickland, Thermalization and isotropization in heavy-ion collisions, arXiv:1312.2285
 - [20] F. Gelis, The initial stages of heavy-ion collisions in the Color Glass Condensate framework, arXiv:1312.5497
 - [21] T. Epelbaum and F. Gelis, Isotropization of the Quark Gluon Plasma, arXiv: 1401.1666
 - [22] S. Mrowczynski, Color collective effects at the early stage of ultrarelativistic heavy-ion collisions Phys. Rev. C VOLUME 49 2191 1994
 - [23] S. Mrowczynski, Chromo-hydrodynamics of the Quark-Gluon Plasma, Nuclear Physics A 785 (2007) 128
 - [24] C. Manuel and S. Mrowczynski, Chromohydrodynamic approach to the unstable quark-gluon plasma, Phys. Rev. D 74, 105003 (2006)
 - [25] B. Schenke, M. Strickland, C. Greiner and M. H. Thoma, Model of the effect of collisions on QCD plasma instabilities, Phys. Rev. D 73, 125004 (2006)
 - [26] S. Mrowczynski and M. H. Thoma, What Do Electromagnetic Plasmas Tell Us about the Quark-Gluon Plasma?, Annu. Rev. Nucl. Part. Sci. 2007. 57:6194
 - [27] M. Mannarelli and C. Manuel, Chromohydrodynamical instabilities induced by relativistic jets, Phys. Rev. D 76, 094007 (2007)
 - [28] M. Attems, A. Rebhan, and M. Strickland, Longitudinal thermalization via the chromo-Weibel instability, 1301.7749
 - [29] M. C. Abraao York, A. Kurkela, E. Lu and G. D. Moore, UV Cascade in Classical Yang-Mills via Kinetic Theory, arXiv:1401.3751
 - [30] E. Calzetta, J. Peralta-Ramos, A hydrodynamic approach to QGP instabilities, arXiv:1309.5412
 - [31] T. Olson and W. Hiscock, Plane Steady Shock Waves in Israel-Stewart Fluids, Ann. Phys. 204, 331 (1990)
 - [32] D. Jou and D. Pavón, Diego, Nonlocal and nonlinear effects in shock waves, Phys. Rev. A44, 6496 (1991)
 - [33] G.S.Denicol, T. Kodama, T. Koide, and Ph. Mota, Shock propagation and stability in causal dissipative hydrodynamics, Phys. Rev. C78, 034901 (2008) (arXiv:0805.1719)
 - [34] I. Bouras, E. Molnár, H. Niemi, Z. Xu, A. El, O. Fochler, F. Lauciello, C. Greiner, and D.H. Rischke, Relativistic shock waves in viscous gluon matter, J.Phys.Conf.Ser.230, 012045 (2010) (arXiv:1004.4615)
 - [35] I. Bouras, E. Molnár, H. Niemi, Z. Xu, A. El, O. Fochler, C. Greiner, and D.H. Rischke, Investigation of shock waves in the relativistic Riemann problem: A comparison of viscous fluid dynamics to kinetic theory, Phys. Rev. C 82, 024910 (2010) (arXiv:1006.0387)

- [36] S. Khlebnikov, M. Kruczenski and G. Michalogiorgakis, Shock waves in strongly coupled plasmas, Phys. Rev. D82, 125003 (2010) (ArXiv: 1004.3803)
- [37] S. Khlebnikov, M. Kruczenski and G. Michalogiorgakis, Shock waves in strongly coupled plasmas II ArXiv: 1105.1355
- [38] I. Bouras, B. Betz, Z. Xu, and C. Greiner, Mach cones in viscous heavy-ion collisions, ArXiv: 1401.3019
- [39] S. Floerchinger and U. A. Wiedemann, Fluctuations around Bjorken flow and the onset of turbulent phenomena, JHEP 11, 100 (2011)
- [40] K. Fukushima, Turbulent pattern formation and diffusion in the early-time dynamics in the relativistic heavy-ion collision. arXiv:1307.1046
- [41] V. Khachatryan, Modified Kolmogorov Wave Turbulence in QCD matched onto Bottom-up Thermalization, Nucl. Phys. A810, 109 (2008) (arXiv:0803.1356)
- [42] M. E. Carrington and A. Rheban, Perturbative and Nonperturbative Kolmogorov Turbulence in a Gluon Plasma, arXiv:1011.0393
- [43] M. Martinez and M. Strickland, Pre-equilibrium dilepton production from an anisotropic quark-gluon plasma, Phys. Rev. C 78, 034917 (2008)
- [44] M. Martinez and M. Strickland, Matching pre-equilibrium dynamics and viscous hydrodynamics, Phys. Rev. C 81, 024906 (2010)
- [45] W. Florkowski, R. Ryblewski and M. Strickland, Testing viscous and anisotropic hydrodynamics in an exactly solvable case, Phys. Rev. C 88, 024903 (2013) (arXiv:1305.7234)
- [46] M. Martinez and M. Strickland, Dissipative dynamics of highly anisotropic systems, Nuclear Physics A 848 (2010) 183197
- [47] M. Martinez and M. Strickland, Non-boost-invariant anisotropic dynamics, Nuclear Physics A 856 (2011) 6887
- [48] M. Strickland, Highly anisotropic dissipative hydrodynamics 1208.2626
- [49] W. Florkowski, M. Martinez, R. Ryblewski and M. Strickland, Anisotropic hydrodynamics basic concepts, 1301.7539
- [50] W. Florkowski, R. Ryblewski and M. Strickland, Anisotropic Hydrodynamics for Rapidly Expanding Systems, Nuclear Physics A 916, 249 (2013) (arXiv:1304.0665)
- [51] M. Strickland Anisotropic Hydrodynamics: Motivation and Methodology, 1401.1188
- [52] Anton Rebhan, M. Strickland and M. Attems, Instabilities of an anisotropically expanding non-Abelian plasma: 1D + 3V discretized hard-loop simulations, Phys. Rev. D 78, 045023 (2008)
- [53] A. Rebhan and D. Steineder, Collective modes and instabilities in anisotropically expanding ultrarelativistic plasmas, Phys. Rev. D 81, 085044 (2010)
- [54] A. Ipp, A. Rebhan and M. Strickland, Non-Abelian plasma instabilities: SU(3) versus SU(2), Phys. Rev. D 84, 056003 (2011)
- [55] M. Attems, A. Rebhan and M. Strickland, Instabilities of an anisotropically expanding non-Abelian plasma: 3D+3V discretized hard-loop simulations, Phys. Rev. D 87, 025010 (2013)
- [56] H. Marrochio, J. Noronha, G. S. Denicol, M. Luzum, S. Jeon and Charles Gale, Solutions of Conformal Israel-Stewart Relativistic Viscous Fluid Dynamics, ArXiv: 1307.6130
- [57] R. Geroch and L. Lindblom, Dissipative relativistic fluid theories of divergence type, Phys. Rev. D 41, 1855 (1990)
- [58] R. Geroch and L. Lindblom, Ann. Phys. (NY) 207, 394 (1991)
- [59] E. Calzetta, Relativistic fluctuating hydrodynamics, Class. Quant. Grav. 15, 653 (1998)
- [60] E. Calzetta and M. Thibeault, Relativistic theories of interacting fields and fluids, Phys. Rev. D 63, 103507 (2001)
- [61] J. Peralta-Ramos and E. Calzetta, Divergence-type nonlinear conformal hydrodynamics, Phys. Rev. D80, 126002 (2009)
- [62] J. Peralta-Ramos and E. Calzetta, Divergence-type theory of conformal fields, Int. J. Mod. Phys. D19, 1721 (2010)
- [63] J. Peralta-Ramos and E. Calzetta, Divergence-type 2+1 dissipative hydrodynamics applied to heavy-ion collisions, Phys. Rev. C82, 054905 (2010)
- [64] P. Lax, *Hyperbolic systems of conservation laws and the mathematical theory of shock waves*, (SIAM, Philadelphia (1973))
- [65] G. B. Nagy and O. A. Reula, A causal statistical family of dissipative divergence-type fluids, J. Phys. A 30, 1695 (1997)
- [66] G. S. Denicol, T. Koide, and D. H. Rischke, Phys. Rev. Lett. 105, 162501 (2010)
- [67] G. S. Denicol, E. Molnár, H. Niemi and D. H. Rischke, Derivation of fluid dynamics from kinetic theory with the 14 moment approximation, Eur. Phys. J. A, 48 11 (2012) 170
- [68] E. Calzetta and J. Peralta-Ramos, Linking the hydrodynamic and kinetic description of a dissipative relativistic conformal theory, Phys.Rev.D82:106003,2010
- [69] J. Peralta-Ramos and E. Calzetta, Macroscopic approximation to relativistic kinetic theory from a nonlinear closure, Phys. Rev. D 87, 034003 (2013)
- [70] W. Hiscock and L. Lindblom, Ann. Phys. 151, 466 (1983)
- [71] W. Hiscock and L. Lindblom, Generic instabilities in first-order dissipative relativistic fluid theories, Phys. Rev. D 31, 725 (1985)
- [72] W. Hiscock and L. Lindblom, Stability in dissipative fluid theories, Contemporary Mathematics 71, 181 (1988).
- [73] T. Olson, Stability and Causality in the Israel-Stewart Energy Frame Theory, Ann. Phys. 199, 18 (1990).
- [74] J. Peralta-Ramos and E. Calzetta, Effective dynamics of a nonabelian plasma out of equilibrium, Phys. Rev. D 86, 125024 (2012)
- [75] E. Calzetta, Non abelian hydrodynamics and heavy ion collisions, arXiv:1311.1845